SOLUTION OF SOME STEADY-STATE PROBLEMS OF HEAT CONDUCTION THEORY FOR ECCENTRIC CYLINDERS WITH A BOUNDARY CONDITION OF THE THIRD KIND

B. A. Vasil'ev

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This paper shows that the steady-state heat conduction problem for eccentric cylinders with a boundary condition of the third kind on one of them can be reduced to the finite-difference solution of the equations. Graphs of the temperature distribution for the case of a plane and cylinder, where the surface of the plane cools down in accordance with Newton's law, are given.

Some steady-state problems of heat conduction theory reduce to a solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ in } D \tag{1}$$

with the boundary condition

$$\frac{\partial u}{\partial n}+hu|_\Gamma=f(p), \tag{2}$$

where h is a positive constant; $f(p)$ is a prescribed function.

In the general case the boundary conditions (2) do not allow effective use of the method of conformal mapping for the solution of the problem.

However, we can distinguish a class of regions for which after transformation to a circle the problem reduces to the solution of certain equations by the method of finite differences [5]. Several physically interesting problems can be solved by using a linear fractional transformation [1]

$$W=\frac{a\xi+b}{c\xi+d}, \tag{3}$$

where $\xi=\rho\exp i\theta$; a, b, c, d are parameters.

The special feature of transformation (3) is that the modulus of the derivative of the image function is also a rational fractional function

$$\left|\frac{dW}{d\xi}\right|=\frac{|ad-cb|}{(c\xi+d)(\bar{c}\bar{\xi}+\bar{d})}, \tag{4}$$

where $\bar{\xi}=\rho\exp(-i\theta)$; this enables us to reduce some of the problems (1), (2) to the solution of second-order linear difference equations.

1. STATEMENT OF PROBLEM AND REDUCTION TO FINITE-DIFFERENCE EQUATION

We consider a system of bipolar coordinates $(\alpha;\beta)$, connected with the Cartesian coordinates (x, y) by the by the relationship [2]

$$x+iy=c\,\mathrm{th}\frac{\alpha+i\beta}{2},\quad \begin{matrix}-\infty<\alpha<+\infty,\\ -\pi\leqslant\beta\leqslant+\pi.\end{matrix} \tag{5}$$

The lines β = const are the arcs of circles passing through points $x=\pm c$

$$x^2+(y-c\,\mathrm{ctg}\,\beta)^2=\frac{c^2}{\sin^2\beta}. \tag{6}$$

The lines α = const are circles orthogonal to them

$$(x-c\,\mathrm{cth}\,\alpha)^2+y^2=\frac{c^2}{\mathrm{sh}^2\alpha}. \tag{7}$$

The Laplace operator and the square of the linear element are written in the form:

$$\Delta T=\frac{(\mathrm{ch}\,\alpha+\cos\beta)^2}{c^2}\left(\frac{\partial^2 T}{\partial\alpha^2}+\frac{\partial^2 T}{\partial\beta^2}\right);$$
$$ds^2=\frac{c^2(d\alpha^2+d\beta^2)}{(\mathrm{ch}\,\alpha+\cos\beta)^2}. \tag{8}$$

Problem 1. To find the steady-state temperature distribution between two infinite eccentric cylinders, one of which (the inner one, $\alpha=\alpha_0$) is maintained at constant temperature, while the surface of the outer cylinder ($\alpha=\alpha_1$) radiates heat according to Newton's law (for a plane $\alpha_1=0$, Fig. 1).

In the selected coordinate system the problem reduces to the solution of the equation

$$\frac{\partial^2 T}{\partial\alpha^2}+\frac{\partial^2 T}{\partial\beta^2}=0,\quad \begin{matrix}\alpha_1<\alpha<\alpha_0,\\ -\pi<\beta<+\pi\end{matrix} \tag{9}$$

with boundary conditions

$$-\frac{\partial T}{\partial\alpha}\frac{\mathrm{ch}\,\alpha+\cos\beta}{c}+hT\Big|_{\alpha=\alpha_1}=0,\quad -\pi\leqslant\beta\leqslant+\pi; \tag{10}$$

$$T|_{\alpha=\alpha_0}=T_0,\qquad -\pi\leqslant\beta\leqslant+\pi, \tag{11}$$

where h is a positive constant.

We seek the solution of problem (9)–(11) in the form

$$T(\alpha;\beta)=T_0+\frac{B_0}{2}(\alpha-\alpha_0)+\sum_{n=1}^{\infty}C_n\,\mathrm{sh}\,n(\alpha-\alpha_0)\cos n\beta. \tag{12}$$

Substituting (12) in the boundary conditions (10) and introducing the symbol

$$B_n=(-1)^n nC_n\,\mathrm{ch}\,n(\alpha_0-\alpha_1),\quad n=1,\ 2,\ 3,\dots, \tag{13}$$

we obtain equations for the coefficients B_n

$$B_{n+1}+B_{n-1}=\left\{2\,\mathrm{ch}\,\alpha_1+2hc\frac{\mathrm{th}\,n(\alpha_0-\alpha_1)}{n}\right\}B_n,\quad n=1,\ 2,\ 3,\ \dots \tag{14}$$

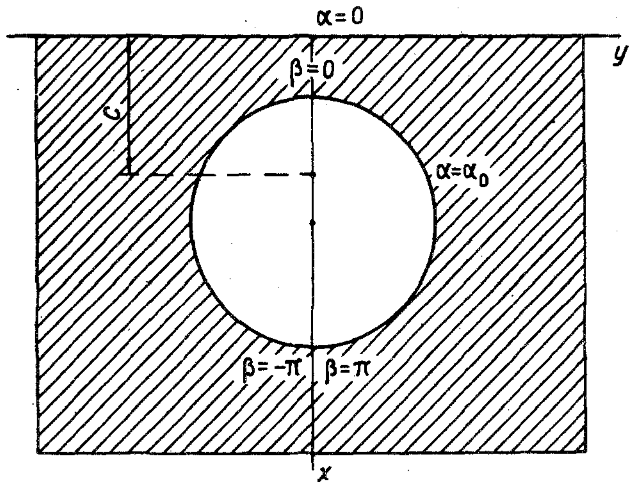

Fig. 1. Cylindrical tube buried in ground.

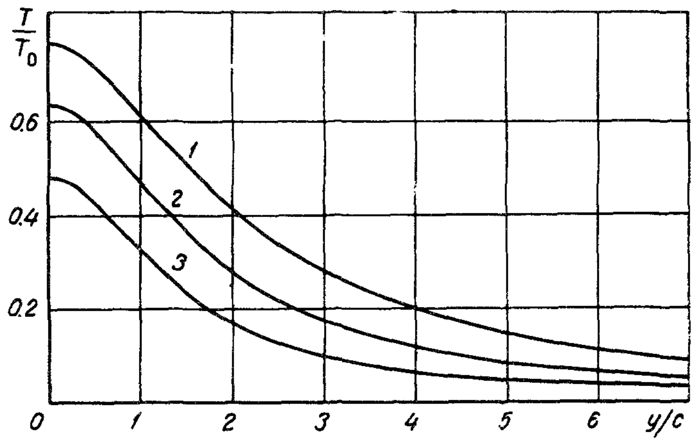

Fig. 2. Temperature as a function of y/c at x = 0 (T_0 is the tube temperature): 1) hc = 0.5; 2) 1; 3) 2.

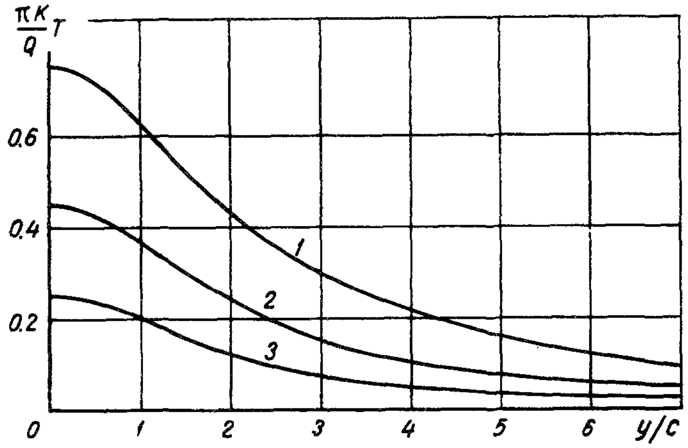

Fig. 3. Temperature as function of y/c at x = 0 (Q is the heat flux from tube) (1–3 as in Fig. 2).

with conditions:

$$B_0[\operatorname{ch}\alpha_1 + hc(\alpha_0 - \alpha_1)] - B_1 = 2hcT_0, \quad (15)$$

$$B_n \to 0,\ n \to \infty. \quad (16)$$

If the solution of equations (14)–(16) is found, the temperature distribution on the surface of the outer cylinder is determined from the formula

$$T|_{\alpha=\alpha_1} = T_0 - \left\{\frac{B_0}{2}(\alpha_0 - \alpha_1) + \right.$$

$$\left. + \sum_{n=1}^{\infty} (-1)^n B_n \frac{\operatorname{th} n(\alpha_0 - \alpha_1)}{n} \cos n\beta \right\}. \quad (17)$$

Problem 2. To find the steady-state temperature distribution between two eccentric cylinders if a uniformly distributed heat flux Q arrives from the surface of the inner cylinder ($\alpha = \alpha_0$), while the outer cylinder ($\alpha = \alpha_1$) radiates heat according to Newton's law. The temperature of the external medium is zero.

Table 1

Solution of Eq. (14) for Case $\alpha_1 = 0$, $\alpha_0 = 1$, hc = 1, $P_0 = 0$, $R_0 = 1$, $C_1 = 1.190618$

n	γ_n	P_n	R_n
1	3.52318	1.00000	0.320201
2	2.96402	3.52318	0.128128
3	2.66336	9.44278	0.0595714
4	2.49966	21.62634	0.0305320
5	2.39996	44.6157	0.0167482
6	2.33334	85.4496	0.00966315
7	2.28571	154.7673	0.00579942
8	2.25000	268.304	0.00359264
9	2.22222	448.917	0.00228402
10	2.20000	729.288	0.00148295

The problem reduces to the solution of the equation

$$\frac{\partial^2 T}{\partial \alpha^2} + \frac{\partial^2 T}{\partial \beta^2} = 0, \quad \begin{array}{l}\alpha_1 < \alpha < \alpha_0, \\ -\pi < \beta < +\pi,\end{array} \quad (18)$$

with boundary conditions

$$-\frac{\partial T}{\partial \alpha}\frac{\operatorname{ch}\alpha + \cos\beta}{c} + hT\Big|_{\alpha=\alpha_1} = 0, \quad -\pi \leqslant \beta \leqslant +\pi, \quad (19)$$

$$\frac{\partial T}{\partial \alpha}\Big|_{\alpha=\alpha_0} = \frac{Q}{\pi K}\frac{\operatorname{sh}\alpha_0}{2\operatorname{ch}\alpha_0 + 2\cos\beta}, \quad -\pi \leqslant \beta \leqslant +\pi, \quad (20)$$

where K is the thermal conductivity, Q is the heat flux in unit time per unit length, and h is a positive constant.

We select a special solution of the problem (18), (20) in the form [2]

$$T = T_1 + T_2, \quad (21)$$

where

$$T_1 = \frac{Q}{\pi K}\left\{\frac{\alpha - \alpha_1}{2} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\exp\{-n\alpha_0\}\operatorname{sh} n(\alpha - \alpha_1)}{\operatorname{ch} n(\alpha_0 - \alpha_1)} \cos n\beta \right\}. \quad (22)$$

Then, to determine T_2 we have the equations:

$$\Delta T_2 = 0 \text{ in } D; \quad \frac{\partial T_2}{\partial n}\Big|_{\alpha=\alpha_0} = 0;$$

$$\frac{\partial T_2}{\partial n} + hT_2\Big|_{\alpha=\alpha_1} = -\frac{\partial T_1}{\partial n}\Big|_{\alpha=\alpha_1}. \quad (23)$$

We seek the solution of problem (23) in the form

$$T_2(\alpha;\ \beta) = \frac{Q}{\pi K}\left\{\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \operatorname{ch} n(\alpha - \alpha_0)\cos n\beta\right\}. \quad (24)$$

Substituting (24), (22), and (21) in the boundary conditions and introducing the symbol

$$M_n = (-1)^{n+1} nA_n \operatorname{sh} n(\alpha_0 - \alpha_1), \quad n = 1, 2, 3, \ldots, \quad (25)$$

we obtain equations for the coefficients M_n

$$M_{n+1} + M_{n-1} = \left\{2\operatorname{ch}\alpha_1 + \right.$$

$$\left. + 2hc\frac{\operatorname{cth}(\alpha_0 - \alpha_1)}{n}\right\} M_n - b_n \quad (26)$$

with conditions

$$M_0 = 0 \text{ and } M_n \to 0,\ n \to \infty, \quad (27)$$

where

$$b_n = a_{n+1} + a_{n-1} - 2\operatorname{ch}\alpha_1 a_n, \quad n = 1, 2, 3, \ldots;$$

$$a_n = \frac{\exp\{-n\alpha_0\}}{\operatorname{ch} n(\alpha_0 - \alpha_1)}, \quad n = 0, 1, 2, 3, \ldots. \quad (28)$$

The coefficient A_0 is determined from the condition

$$hcA_0 = \operatorname{ch}\alpha_1 - a_1 - M_1. \quad (29)$$

If the solution of Eqs. (26), (27), and (29) is found, the temperature distribution on the outer cylinder can be written in the form

$$T\Big|_{\alpha=\alpha_1} = \frac{Q}{\pi K}\left\{\frac{A_0}{2} + \right.$$

$$\left. + \sum_{n=1}^{\infty} (-1)^{n+1} M_n \frac{\operatorname{cth} n(\alpha_0 - \alpha_1)}{n}\cos n\beta\right\}. \quad (30)$$

2. SOLUTION OF EQUATIONS IN FINITE DIFFERENCES

We consider the possible solutions of Eq. (14), which we write in the form

$$B_{n+1} + B_{n-1} = \gamma_n B_n, \quad n = 1, 2, 3, \ldots, \quad (31)$$

where

$$\gamma_n = 2\operatorname{ch}\alpha_1 + 2hc\frac{\operatorname{th} n(\alpha_0 - \alpha_1)}{n}.$$

As follows from the theory of continued fractions [4], two linearly independent solutions of Eq. (31) can be represented as the numerator and denominator of the n-th convergent fraction

$$\frac{P_{n+1}}{Q_{n+1}} = \gamma_1 - \frac{1}{\gamma_2} - \ldots \frac{1}{\gamma_n}, \quad n = 1, 2, 3, \ldots. \quad (32)$$

The numbers P_n can be obtained from the equation

$$P_{n+1} + P_{n-1} = \gamma_n P_n, \quad n = 1, 2, 3, \ldots \quad (33)$$

Table 2

Solution of Eq. (26) for Case $\alpha_1 = 0$; $\alpha_0 = 1$, hc = 1, $P_0 = 0$, $R_0 = 1$, $M_0 = 0$

n	γ_n	P_n	R_n	b_n	M_n
1	4.62608	1.00000	0.235984	0.55917	0.148893
2	3.03732	4.62608	0.0916843	0.17140	0.129625
3	2.66998	13.05088	0.0424913	0.026752	0.0734212
4	2.50034	30.21951	0.0217659	0.0036946	0.0369549
5	2.40004	62.5081	0.0119308	0.00050142	0.0220346
6	2.33333	119.8024	0.00686848	0.00006788	0.0127277
7	2.28571	217.0304	0.00409564	0.0000091881	0.00759544
8	2.25000	376.266	0.00249296	0.0000012433	0.00462405
9	2.22222	629.568	0.00151352	0.0000001683	0.00280744
10	2.20000	1022.773	0.00087041	—	0.00161454

and the "initial" conditions

$$P_0 = 0;\ P_1 = 1. \tag{34}$$

Then

$$R_2 = \gamma_1;\ P_3 = \gamma_1\gamma_2 - 1;\ P_4 = \gamma_1\gamma_2\gamma_3 - \gamma_1 - \gamma_3\ \ldots \tag{35}$$

For γ_n given by expressions (31) or (26) the numbers P_n are a monotonically and indefinitely increasing sequence of numbers [4].

To construct the monotonically decreasing, zero-bounded sequence of numbers R_n satisfying the equation

$$R_{n+1} + R_{n-1} = \gamma_n R_n,\quad n = 1,\ 2,\ 3,\ \ldots \tag{36}$$

if

$$R_n \to 0,\quad n \to \infty, \tag{37}$$

we use the main properties of linear equations in finite differences [3]. We multiply Eqs. (33) and (36) by R_n and P_n, respectively, and substract one from the other. After transformations we obtain

$$\begin{vmatrix} R_n & P_n \\ R_{n+1} & P_{n+1} \end{vmatrix} = P_{n+1}R_n - R_{n+1}P_n =$$

$$= P_n R_{n-1} - R_n P_{n-1} = \text{const} = 1, \tag{38}$$

from which

$$\left(\frac{R_n}{P_n}\right) = \frac{R_{n+1}}{P_{n+1}} - \frac{R_n}{P_n} =$$

$$= \frac{R_{n+1}P_n - P_{n+1}R_n}{P_n P_{n+1}} = -\frac{1}{P_n P_{n+1}} \tag{39}$$

or*

$$R_n = P_n \sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}}. \tag{40}$$

From (40) we have

$$R_0 = 1;\ R_n \to 0,\quad n \to \infty. \tag{41}$$

*It is easy to show that the numbers R_n differ by a constant factor in the case of different "initial" conditions (34) for P_n.

Thus, the general solution of Eq. (31) can be written in the form

$$B_n = C_1 R_n + C_2 P_n,\quad n = 0,\ 1,\ 2,\ \ldots, \tag{42}$$

where C_1 and C_2 are arbitrary constants.

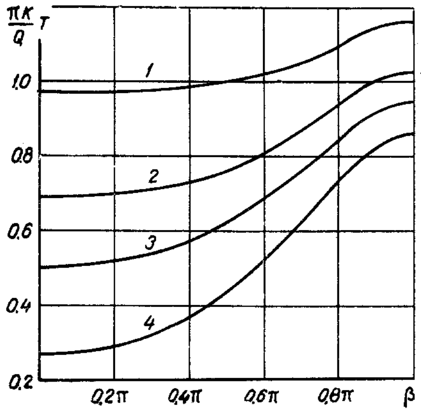

Fig. 4. Temperature distribution on surface of tube (Q is the heat flux from the tube): 1) hc = 0.5; 2) 1; 3) 2; 4) ∞.

From the condition of boundedness of the numbers B_n we must put $C_2 = 0$ and then we have

$$B_n = C_1 R_n,\quad n = 0,\ 1,\ 2,\ \ldots. \tag{43}$$

Substituting (43) into condition (15), we obtain

$$C_1 = \frac{2hcT_0}{R_0[\operatorname{ch}\alpha_1 + hc(\alpha_0 - \alpha_1)] - R_1} \tag{44}$$

or, finally,

$$B_n = \frac{2hcT_0 R_n}{R_0[\operatorname{ch}\alpha_1 + hc(\alpha_0 - \alpha_1)] - R_1},\quad n = 0,\ 1,\ 2,\ \ldots. \tag{45}$$

Table 1 gives the values of the numbers γ_n, P_n, and R_n for the case $\alpha_1 = 0$; $\alpha_0 = 1$; hc = 1.

Figure 2 shows graphs of the temperature distribution on the plane ($\alpha_1 = 0$) for cases $\alpha_0 = 1$; hc = 0.5; hc = 1; hc = 2.

As follows from (26), the solution of problem 2 reduces to the solution of the equation

$$M_{n+1} + M_{n-1} = \gamma_n M_n - b_n,\quad n = 1,\ 2,\ 3,\ \ldots \tag{46}$$

with the conditions

$$M_0 = 0; \quad M_n \to 0, \quad n \to \infty, \tag{47}$$

where

$$\gamma_n = 2\,\mathrm{ch}\,\alpha_1 + 2hc\frac{\mathrm{cth}\, n(\alpha_0 - \alpha_1)}{n},$$
$$n = 1, 2, 3, \ldots. \tag{48}$$

The general solution of Eq. (46) can be written in the form [3]

$$M_n = \overline{M}_n + C_1 R_n + C_2 P_n, \tag{49}$$

where R_n and P_n are the solutions of the corresponding homogeneous equation; C_1 and C_2 are arbitrary constants; $\overline{M}_n$ is a special solution of the inhomogeneous equation.

The solutions of R_n and P_n can be constructed similarly to (35), (40) from the numbers (48). To construct a special solution we can use the method of variation of constants [3]

$$\overline{M}_n = C_1(n) R_n + C_2(n) P_n. \tag{50}$$

After simple transformations we obtain

$$C_1(n) = \sum_{k=1}^{n} b_k P_k; \quad C_2(n) = \sum_{k=n+1}^{\infty} b_k R_k. \tag{51}$$

The solution of Eq. (46) with conditions (47) can finally be written in the form

$$M_n = R_n \sum_{k=1}^{n} b_k P_k + P_n \sum_{k=n+1}^{\infty} b_k R_k. \tag{52}$$

Table 2 gives the values of the numbers γ_n; P_n; R_n; b_n; M_n for the case $\alpha_1 = 0$; $\alpha_0 = 1$; hc = 1.

Figure 3 shows graphs of the temperature distribution on the plane ($\alpha_1 = 0$) for cases $\alpha_0 = 1$; hc = 0.5; hc = 1; hc = 2. Figure 4 shows graphs of the temperature distribution on the tube surface ($\alpha = \alpha_0 = 1$) for cases $\alpha_1 = 0$; hc = 0.5; hc = 1; hc = 2.

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Kalinin Polytechnic Institute, Leningrad